Theorem Let $B = (B_t)_t \le 1$ be standard 1-dimensional BM and let $W = B + f$ where $f \in C[0,1]$ with $f(0) = 0$. Let μ and ν be the distributions of B and W respectively. Then,

(1) If f is absolutely continuous and $f' \in L^2[0,1]$, then μ and ν are mutually absolutely continuous and

$$\frac{d\nu}{d\mu} = \exp\left\{ \int_0^1 f'\, dB \;-\; \frac{1}{2}\int_0^1 (f')^2 \right\}.$$

(2) If f is not as in (1), then $\mu \perp \nu$.

Proof

(1) Let $\alpha_{n,k} = \int_0^1 f'\, h_{n,k}$ where $h_{n,k}$ are the Haar functions. Let $\xi_{n,k}$ be i.i.d. N(0,1) and let $\eta_{n,k} = \xi_{n,k} + \alpha_{n,k}$. Let $\tilde{\mu}, \tilde{\nu}$ denote the distributions of $(\xi_{n,k})$ and $(\eta_{n,k})$ (measures on $(\mathbb{R}^\infty, \mathcal{B}_{\mathbb{R}^\infty})$). Since $\|\alpha\|_{\ell_2} = \|f'\|_{L^2[0,1]}$, it follows from the earlier theorem that $\tilde{\mu}$ and $\tilde{\nu}$ are mutually absolutely continuous and

$$\frac{d\tilde{\nu}}{d\tilde{\mu}}(\omega) = \exp\left\{ \sum \alpha_{n,k}\, \omega_{n,k} \;-\; \frac{1}{2}\sum \alpha_{n,k}^2 \right\}.$$

Note that we may construct B and W as the same transformation applied to the sequences $(\xi_{n,k})$ and $(\eta_{n,k})$ as

$$B(t) = \sum \xi_{n,k} \int_0^t h_{n,k} \qquad W(t) = \sum \eta_{n,k} \int_0^t h_{n,k}$$

which shows that μ and ν are mutually absolutely continuous and the claimed formula for the Radon-Nikodym derivative also follows, since $\int f'\, dB = \sum \alpha_{n,k}\xi_{n,k}$ and $\int (f')^2 = \sum \alpha_{n,k}^2$. [Question: Where did we use $f(0) = 0$? Otherwise the result ought to be false!]

(2) Let $\{\psi_n\}$ be an ONB for $L^2[0,1]$ with the property that each ψ_n is smooth and $\psi_n(0) = \psi_n(1) = 0$. Define $\alpha_n = \int f\, \psi_n'$.

$$\textbf{Claim:} \qquad \sum \alpha_n^2 = \infty.$$

Assuming the claim, consider $X_n := \int B_t \psi_n'(t) dt$ and $Y_n := \int W_t \psi_n'(t) dt = X_n + \alpha_n$. Note that X_n is just $\int \psi_n(t)\, dB(t)$ and hence X_n are i.i.d. $N(0,1)$. Thus by our earlier theorem the distributions of $X = (X_n)_n$ and $Y = (Y_n)_n$ are singular. As X and Y are functions of B and W, it follows that μ and ν must be singular too.

It just remains to prove the claim. For any finite linear combination $g = \sum_{k=1}^n c_k \psi_k$, define $\mathcal{L}[g] := \int g' f = \sum_{k=1}^n c_k \alpha_k$. If the claim was not true, we would get $|\mathcal{L}[g]| \le \sqrt{\sum c_k^2}\sqrt{\sum_{k=1}^n \alpha_k^2} \le \|g\|_{L^2}\|\alpha\|_{\ell_2}$. This shows that $\mathcal{L}$ extends to a bounded linear functional on $L^2[0,1]$ and hence it can be represented in the form $\mathcal{L}[g] = \int g\, h$ for some $h \in L^2[0,1]$. Writing $H(t) = \int_0^t h$, we can rewrite this as follows for any g that is a finite linear combination of ψ_ns.

$$\int g' f = \mathcal{L}[g] = \int g\, h = \int g H' = g.H\big|_0^1 - \int g' H = -\int g' H$$

as $g(0) = 0 = g(1)$. From this, it is clear that $f(t) = H(t) = \int_0^t h$ which contradicts the assumption that f does not have an L^2-derivarive.

□

Let B be a 1-dimensional standard BM. Write $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. Then for any $0 = t_0 < t_1 < \ldots \le t_n \le 1$, the density of $(B(t_1), \ldots, B(t_n))$ can be written as

$$P_{t_1,\ldots,t_n}(x_1, \ldots x_n) = \prod_{k=1}^{n} p_{t_k - t_{k-1}}(x_k - x_{k-1}).$$

Hence, for any $t_1 < \ldots < t_n < 1$, the conditional density of $(B(t_1), \ldots, B(t_n))$ given $B(1) = 0$ is seen to be

$$Q_{t_1,\ldots,t_n}(x_1, \ldots x_n) = \frac{P_{t_1,\ldots,t_n,1}(x_1, \ldots x_n, 0)}{p_1(0)}.$$

Definition Brownian bridge is a $C[0,1]$ valued random variable $X = (X_t)_{0 \le t \le 1}$ such that $X_0 = X_1 = 0$ a.s. and whose finite dimensional distributions are given by the densities $Q_{t_1,\ldots,t_n}(\cdot)$.

A natural question is whether the Brownian bridge exists. We have already done the work for this in construction Brownian motion, and the following exercise shows the existence as well as useful ways of representing Brownian bridge in terms of Brownian motion and vice versa.

Exercise

(1) Let B be a standard 1-dimensional BM. Define $W_t = B_t - tB_1$ and $X_t = (1-t)B\left(\frac{t}{1-t}\right)$ for $t < 1$ and $X_1 = 0$. Show that X and W are Brownian bridges (and hence $X \stackrel{d}{=} W$).

(2) Given a Brownian bridge W and $\xi \sim N(0,1)$ independent of W, define $Y_t = W_t + t\,\xi$ for $0 \le t \le 1$. Show that Y is a standard BM.

The following exercise shows that almost sure local properties of Brownian bridge are the same as that of Brownian motion. For example Brownian bridge paths are Hölder continuous of any order less than $1/2$ but nowhere Hölder of order greater than $1/2$; the bridge crosses zero infinitely may times in any vicinity of $t = 0$; the zero set has dimension 1/2; the graph has dimension 3/2 etc.

Exercise Let B be a standard BM and let W be a standard Brownian bridge. Show for any fixed $T < 1$ that the distributions of $(W_t)_{t \le T}$ and $(B_t)_{t \le T}$ are mutually absolutely continuous.

An occurence of Brownian bridge Let $X_1, X_2, \ldots$ be i.i.d. random variables from a distribution with distribution function F on the real line. Then $F_n(x) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{X_k \le x}$ is called the n^{th} empirical distribution function (it is a random distribution function). By the law of large numbers $F_n \to F$ pointwise. How big are the differences between F_n and F? As the difference is going to zero, we must scale by the right amount, which turns out to be $\sqrt{n}$. We state the following result without proof.

Result Let $h(t) = F^{-1}(t)$ so that $F(h(t)) = t$ for all t. Here F^{-1} is the right-continuous inverse of F (or just assume that F is continuous). Then

$$\left(\sqrt{n}\,[F_n(h(t)) - t)]\right)_{t \le 1} \stackrel{d}{\to} \text{Brownian bridge.}$$

(If you really think about this it will raise some questions about the meaning of the statement itself such as in what space is the convergence in distribution taking place?)